Cayley Graph and Metric Spaces

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1 Recap

- Group as Symmetries of Graph: eg: $D_n, \mathbb{Z}, \mathbb{Z}^2$
- Group Homomorphism and Normal Subgroup: homomorphism definition, normal subgroup, first isomorphism theorem.
- Group Presentation: Every group is a quotient of a free group.

2 Introduction

- Group actions: definition, Cayley's theorem.
- Graph: definition, graph isomorphisms, graph automorphisms, group action on graphs.
- Cayley Graphs: definition, automorphisms of Cayley graph, Cayley graph for free group F_2.
- Metric Spaces: definition, path metric on a graph, geometric realization of graph, groups as metric spaces, isomorphisms between metric spaces, isometry of metric spaces.
- Group Action on Metric Spaces

3 Group Actions

Definition: A group action of a group G on a set X is a mapping:

$$G \times X \to X$$

such that

$$1 \cdot x = x, \forall x \in X$$

$$h \cdot (g \cdot x) = (hg) \cdot x$$

A group can act on itself in two ways:

$$\text{left multiplication: } g \cdot x = gx, \forall x \in G.$$

$$\text{conjugation: } g \cdot x = gxg^{-1}, \forall x \in G$$

Theorem (Cayley): Every finite group is isomorphic to a subgroup of some permutation group. The theorem follows from the fact that the left multiplication operator permutes group G and that $g \to \Phi_g$ is a homomorphism.

4 Graphs

A graph is a combinatorial object defined as follows: there is a nonempty set V and a set E, and a function from E to the set of unordered pairs of elements in V. This function is called the endpoint function.

Definition (Graph isomorphism): An isomorphism between graph (V, E) and (V', E') is a pair of bijective mappings $\phi_1 : V \to V'$ and $\phi_2 : E \to E'$ such that the endpoint function is respected.

Definition (Graph Automorphism): A graph automorphism is an isomorphism from a graph to itself. Graph automorphisms are also called symmetries of graphs.
To say that every group is a group of symmetry of some graph, we mean that for every group G, there exists some graph such that the automorphism group of the graph is isomorphic to the group.

Examples of Graph Automorphism: Cube under rotation; Infinite grid under translation; automorphisms of $K_{n,n}$ ($L_n \leftrightarrow R_n$; Permutation of left vertices and right vertices).

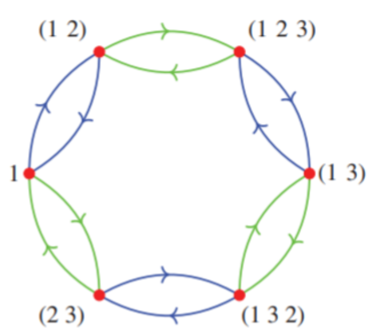
5 Cayley Graph

Definition: The Cayley graph for G wrt one of its generating set S, is a directed, labeled graph $\Gamma(G, S)$ given as follows: the vertex set is G, there is a directed edge from g to gs,$\forall g \in G$ and $s \in S$, and the edge connecting g and gs is labeled as s.

Theorem: G is a group. Then the automorphism group of its Cayley graph, $(\Gamma(G, S))$ is isomorphic to G.

On the one hand, Φ_g(left-multiply each vertex and sends edge $h \to hs$ to $gh \to ghs$) is an automorphism because it is bijective, label-presering and respects end-point function. On the other hand, suppose Φ is an automorphism that sends 1 to g, then we can show that $\Phi = \Phi_g$. Firstly, the two mappings agree for set S. By induction we can show that they agree on all vertices. Since they are both automorphisms, the fact that they agree on all vertices implies that they respects end-point function and preserves edge label.

Examples of Cayley Graph: Cayley graph for S_3 generated by $\{(1, 2), (2, 3)\}$ is as follows:



The Cayley graph for F_2 is a tree.

6 Metric Spaces

Definition: A metric space X is a set and a function

$$d : X \times X \to X$$

satisfying:

$$d(x, y) \geq 0; d(x, y) = 0 \text{ iff } x = y$$

$$d(x, y) = d(y, x)$$

$$d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X$$

Examples of metric spaces: In $\mathbb{R}^n$, we have L^p distances: $\|x - y\| = (\sum_1^n |x_i - y_i|^p)^{\frac{1}{p}}$. ($L^1$ distance is aka Manhattan Distance). In a graph, we can define *path metric*: the distance between any two points is defined as the length of the shortest edge path. if we want to make the measure space "continuous," then we arrive at the geometric realization of X—we allow points to lie on an edge and identify each edge with $[0, 1]$.

Group as metric spaces: a.k.a. word metric, the metric coincides with path metric of its corresponding Cayley graph. "A group together with a generating set is a metric space."

Isometries between metric spaces: Length-preserving bijection between two spaces.

Isometries of metric spaces: Length-preserving bijection from a metric space X to itself. We denote the isometry group of a group X by $\mathrm{Isom}(X)$.
Group Actions on metric spaces: A group action on a metric space X is a mapping

$$G \times X \to X$$

such that

$$1 \cdot x = x, \forall x \in X$$

$$g \cdot (h \cdot x) = (gh) \cdot x, \forall g, h \in G, \forall x \in X. \text{ (Associativity)}$$

$$d(g \cdot x, g \cdot y) = d(x, y), \forall g \in G, \forall x, y \in X. \text{ (Distance-Preserving)}$$

Connection between geometric and algebraic properties of groups: We show an example illustrating how can we exploit geometric properties of a group to obtain purely algebraic information about the group. We say a group is "torsion free" if it does not contain any element of finite order except for the identity. A group acts freely on a set X if $\{g \in G | \exists p \in G, g \cdot p = p\} = \{1\}$. We will prove that if G has a free action by isometries on $\mathbb{R}^n$ with Euclidean metric, then G is torsion free.

The proof goes as follows. Suppose g is of finite order, and w is a point in $\mathbb{R}^n$, then $\langle g \rangle w$ is a finite set. Assume the centriod of the set is a (Recall that the centriod of a finite set $\{x_1, \ldots, x_n\}$is the point y that minimizes $\sum_i^n d(y, x_i)$; The uniqueness of centriod follows either from physical intuition or from the fact that the sum of distance is convex for y.) Then the centriod of $g \cdot \langle g \rangle a$ is $g \cdot a$, because g is distance-preserving. Thus we have $a = g \cdot a$, and thus $g = 1$. Since the only element of finite order is identity, we proved that G is torsion-free.